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The action of the Steenrod algebra on Tate cohomology

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Abstract

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Let G be a finite group and k a field of characteristic p dividing $|G|$. In this paper, we examine the action of the Steenrod operations on the Tate cohomology $\hat{H}^*(G, k)$. In particular, we prove that all Steenrod operations from negative to positive degree Tate cohomology vanish if and only if all products in negative degree Tate cohomology vanish.

1. Introduction

Let G be a finite group and k a field of characteristic p dividing $|G|$. A recent paper [1] examined the question of when there can be non-zero products between elements of negative degree in Tate cohomology $\hat{H}^-(G, k)$. It was observed that if there is such a non-zero product, then there is an element $y \in H^*(G, k)$ of positive degree, such that the Krull dimension of $H^*(G, k)/\mathrm{Ann}(y)$ is equal to one.

Our aim in this paper is to relate the above question to the action of the Steenrod algebra $\mathscr{A}$ on Tate cohomology. Namely, we have a short exact sequence

$$0 \to H^*(G, \mathbb{F}_p) \to \hat{H}^*(G, \mathbb{F}_p) \to \hat{H}^-(G, \mathbb{F}_p) \to 0 .$$

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The statement that this splits as a sequence of $H^*(G, \mathbb{F}_p)$-modules is equivalent to the statement that all products on $\hat{H}^-(G, \mathbb{F}_p)$ are zero [1, Lemma 2.1]. On the other hand, the statement that this splits as a sequence of $\mathscr{A}$-modules is equivalent to the statement that there are no non-zero Steenrod operations going from negative to positive degree. Our main theorem states that the above sequence splits as $H^*(G, \mathbb{F}_p)$-modules if and only if it splits as $\mathscr{A}$-modules.

Theorem 1.1. *Let G be a finite group and p a prime dividing $|G|$. Then the following are equivalent:*

(i) *All products of elements of negative degree Tate cohomology $\hat{H}^-(G, \mathbb{F}_p)$ vanish.*

(ii) *All Steenrod operations from negative degree to positive degree on $\hat{H}^*(G, \mathbb{F}_p)$ are zero.*

Corollary 1.2. *If there is a non-zero Steenrod operation from negative degree to positive degree on $\hat{H}^*(G, \mathbb{F}_p)$, then there is an element $y \in H^*(G, \mathbb{F}_p)$ such that the Krull dimension of $H^*(G, k)/\mathrm{Ann}(y)$ is equal to one.* □

Corollary 1.3. *If the centre of a Sylow p-subgroup of G has rank at least two, then all Steenrod operations from negative degree to positive degree on $\hat{H}^*(G, \mathbb{F}_p)$ are zero.* □

Corollary 1.4. *Suppose that G is a finite group of p-rank two. Then $H^*(G, \mathbb{F}_p)$ is Cohen–Macaulay if and only if all Steenrod operations from negative degree to positive degree on $\hat{H}^*(G, \mathbb{F}_p)$ are zero.* □

We end the Introduction with a few remarks. First, we remark that in the case where G is cyclic of order p, all Steenrod reduced powers are non-zero on a class of degree -1, so that the extension of $\mathscr{A}$-modules is really very far from split in this case.

The next remark is that in general, one can deduce the action of $\mathscr{A}$ on Tate cohomology from the action on ordinary cohomology as follows. We have noted that we only have an extension problem, so it remains to calculate the effect of P^n when it goes from negative to positive degrees. Specifically, if $x \in \hat{H}^-(G, \mathbb{F}_p)$ and $P^n(x) \in \hat{H}^+(G, \mathbb{F}_p)$, we choose a non zero-divisor $\zeta \in H^*(G, \mathbb{F}_p)$ (Lemma 3.1) and a value of r large enough so that $\zeta^r x \in \hat{H}^+(G, \mathbb{F}_p)$. We may suppose by induction on n that $P^i x$ is known for $i < n$, and the Cartan formula gives $P^n(\zeta^r x) = \zeta^r P^n(x) +$ lower terms which are known by induction. Since multiplication by ζ^r is injective in positive degrees, this determines $P^n(x)$. The proof of the main theorem is a development of this idea.

Finally, we remark that in the case where G is elementary abelian of rank bigger than one, Corollary 1.2 shows that the Steenrod operations are all trivial from negative to positive degrees, which answers a question of J. Morava.

2. Steenrod operations

We begin with a brief discussion of how the Steenrod algebra $\mathscr{A}$ acts on Tate cohomology.

It seems most efficient to use a little equivariant topology. Specifically, we shall use the fact that there is a good homotopy category of G-spectra [5] in which equivariant cohomology theories $F_G^*(-)$ are represented, in the sense that there is a G-spectrum F such that for any based G-space X we have $F_G^*(X) = [X, F]_G^*$. Thus ordinary cohomology of the Borel construction $H^*(EG \times_G X; \mathbb{F}_p)$ is represented by a spectrum b; in particular the cohomology of a point is the group cohomology $b^* = H^*(G, \mathbb{F}_p)$. The relevance of this is that Tate–Swan cohomology of G-spaces can be represented by the spectrum $t = b \wedge \tilde{E}G$, where $\tilde{E}G$ denotes the unreduced suspension of EG [3]; in particular we have $t^* = \hat{H}^*(G, \mathbb{F}_p)$. Now b is a commutative ring spectrum in the sense that there are maps $\eta : S^0 \to b$ and $\mu : b \wedge b \to b$ which make the unit and associativity diagrams commute; for our purposes the best way to see this is that b may be described as a function spectrum of maps into a well-known ring spectrum.

Proposition 2.1. *There is an equivalence $b = F(EG_+, H\mathbb{F}_p)$ where $H\mathbb{F}_p$ is the nonequivariant spectrum representing ordinary* mod p *cohomology, regarded as a G-spectrum as in Definition* II.1.1 *of* [5].

Proof. For a pointed G-space X we have

$$[X, F(EG_+, H\mathbb{F}_p)]_G^* = [EG_+ \wedge X, H\mathbb{F}_p]_G^* = [EG_+ \wedge_G X, H\mathbb{F}_p]^* ,$$

where the first equality is formal, and the second is a familiar change of groups isomorphism [5, Theorem II.4.5], valid since EG_+ is free and $H\mathbb{F}_p$ is G-fixed. The lemma then follows from uniqueness of representing objects. □

It is then obvious that t is a ring spectrum since $\tilde{E}G \wedge \tilde{E}G \simeq \tilde{E}G$, and hence it is a b algebra.

Corollary 2.2. *The action of $\mathscr{A}$ on Tate cohomology satisfies the Cartan formula.*

Proof. It is familiar that $\mathscr{A} = [H\mathbb{F}_p, H\mathbb{F}_p]^*$ by the Yoneda lemma, so it follows from Proposition 2.1 that there is a map $\mathscr{A} \to [b, b]_G^*$. Indeed in [4] it is shown that with the coefficient ring b^* it gives all stable operations: $b^* \otimes \mathscr{A} \cong [b, b]_G^*$.

Now, given any operation $\theta \in [b, b]_G^* = b^*b$ we may consider $\mu^*(\theta) \in b^*(b \wedge b)$. Again, it is shown in [4] that $b^*(b \wedge b) \cong b^*b \otimes_{b^*} b^*b$, and it follows that there is a diagram

$$\begin{array}{ccc} b \wedge b & \xrightarrow{\mu} & b \\ {\scriptstyle \sum_i \theta_i' \wedge \theta_i''} \downarrow & & \downarrow {\scriptstyle \theta} \\ b \wedge b & \xrightarrow{\mu} & b \end{array}$$

Using this and the fact that t is a b algebra we may construct a diagram which shows for $x, y \in t^*$ that $\theta(xy) = \sum_i \theta_i'(x)\theta_i''(y)$. Using Proposition 2.1 or results from [4] it follows that if $\theta \in \mathcal{A}$ then θ_i' and θ_i'' are the usual terms in the Cartan formula. □

The other property that we shall need is that an element $\theta \in \mathcal{A}$ should act in dual ways on the positive and negative parts of the Tate cohomology. First we must explain that homology theories are also represented in the category of G-spectra, and in particular the homology of the Borel construction $H_*(EG \times_G X; \mathbb{F}_p)$ is represented by the coBorel spectrum $c = b \wedge EG_+$, which is visibly a b module spectrum. Evidently the coefficient groups of c are the homology groups of G in the sense that $c^{-i} = c_i = H_i(G, \mathbb{F}_p)$, and, as might be expected, the action of b^* on c_* corresponds to the action of $H^*(G)$ on $H_*(G)$ by cap product. This may be proved with a little bit of duality theory for finite free G-complexes, using the fact that if X is finite and free $D(X/G) = (D_G X)/G$ in an appropriate sense [5, Proposition III.2.12]. Furthermore, the evident cofibre sequence $b \to t \to \Sigma c$ includes the short exact sequence under consideration as its sequence of homotopy groups; it is obviously a sequence of b^*b modules.

Proposition 2.3. (a) *The action of θ on Borel cohomology corresponds to the usual action under the isomorphism $b^*(X) = H^*(EG \times_G X; \mathbb{F}_p)$.*

(b) *The action of θ on coBorel homology corresponds to the usual action under the isomorphism $c_*(X) = H_*(EG \times_G X; \mathbb{F}_p)$.*

Proof. Part (a) is clear from Proposition 2.1 and part (b) follows by duality. □

Corollary 2.4. *The action of $\mathcal{A}$ on $\hat{H}^*(G, \mathbb{F}_p)$ obeys Tate duality in the sense that if $a \geq 0$ the maps $\theta : \hat{H}^a(G, \mathbb{F}_p) \to \hat{H}^{a+n}(G, \mathbb{F}_p)$ and $\theta : \hat{H}^{-a-n-1}(G, \mathbb{F}_p) \to \hat{H}^{-a-1}(G, \mathbb{F}_p)$ are duals.*

Proof. This follows from Proposition 2.3 since the action of $\mathcal{A}$ on homology is defined to be dual to that on cohomology. □

3. Proof of the main theorem

We begin with a lemma describing properties of the ring structure needed in the proof.

Lemma 3.1. (i) (Duflot [2]) *$H^*(G, \mathbb{F}_p)$ has depth at least one. In other words, there is an element ζ of positive degree in $\hat{H}^*(G, \mathbb{F}_p)$ with the property that multiplication by ζ is injective in positive degrees and surjective in negative degrees.*

(ii) *There exists a pair of elements of $\hat{H}^-(G, \mathbb{F}_p)$ whose product is non-zero, if and only if there exists $x \in \hat{H}^-(G, \mathbb{F}_p)$ with the property that for $r > 0$ large enough, $\zeta^r x$ is a non-zero element of positive degree in $\hat{H}^*(G, \mathbb{F}_p)$.*

Proof. Since (i) is proved in [2], we shall only prove (ii). By [1, Lemma 2.1], there exists a pair of elements of $\hat{H}^-(G, \mathbb{F}_p)$ with non-zero product, if and only if there are elements $u \in \hat{H}^-(G, \mathbb{F}_p)$ and $v \in \hat{H}^+(G, \mathbb{F}_p)$ with uv a non-zero element of positive degree. If the latter holds, choose $r > 0$ large enough so that $\deg(\zeta^r) > \deg(uv)$. Since multiplication by ζ^r is surjective on $\hat{H}^-(G, \mathbb{F}_p)$, we can choose $w \in \hat{H}^-(G, \mathbb{F}_p)$ with $u = \zeta^r w$. Set $x = wv$, so that $\deg(x) < 0$ by choice of r. Then $\zeta^r x = \zeta^r wv = uv$ is a non-zero element of positive degree. □

We begin the proof of the main theorem by assuming that all products in $\hat{H}^-(G, \mathbb{F}_p)$ are zero, and proving by induction on n that for $x \in \hat{H}^-(G, \mathbb{F}_p)$ with $\deg(P^n(x)) \geq 0$, $P^n(x) = 0$ (if $p = 2$, replace P^n by Sq^n). Suppose, to the contrary, that $P^n(x) = y$ is a non-zero element of positive degree in $\hat{H}^*(G, \mathbb{F}_p)$. Let $\zeta \in H^*(G, \mathbb{F}_p)$ be as in part (i) of the lemma, and choose $r > 0$ large enough so that $\deg(\zeta^r) + \deg(x) > 0$. Since $\zeta^r y \neq 0$, we can choose, by Tate duality, an element z in negative degree so that $\zeta^r yz$ is non-zero in $\hat{H}^{-1}(G, {}_p)$. Thus $yz \neq 0$, but $xz = 0$. But now the Cartan formula gives

$$0 = P^n(xz) = yz + \sum_{j=1}^{n} P^{n-j}(x) P^j(z) .$$

The terms in the sum with $\deg(P^{n-j}(x)) \geq 0$ vanish by induction, while the remaining terms are products of negative degree elements and so they vanish by hypothesis. This means that $yz = 0$, and this contradiction completes the induction. We also need to prove that the Bockstein β vanishes on $\hat{H}^{-1}(G, \mathbb{F}_p)$, but of course this happens whenever $|G|$ is divisible by p^2.

Conversely, suppose that there is a non-zero product in $\hat{H}^-(G, \mathbb{F}_p)$, so that by part (ii) of the lemma, we can choose $x \in \hat{H}^{-m}(G, \mathbb{F}_p)$ with $x\zeta^r$ a non-zero element of positive degree in $\hat{H}^*(G, \mathbb{F}_p)$. Let $\deg(\zeta^r) = 2n$, so that $0 < m < 2n$. Note also that ζ is a non-zero divisor in positive degrees, so that $x\zeta^{pr} \neq 0$. Choose $z \in \hat{H}^{-2pn+m-1}(G, \mathbb{F}_p)$ with $zx\zeta^{pr} \neq 0$ in $\hat{H}^{-1}(G, \mathbb{F}_p)$. Since $P^n(\zeta^r) = \zeta^{pr}$, it follows by Tate duality that $P^n(zx)$ is an element of negative degree whose product with ζ^r is non-zero, so that in particular $P^n(zx) \neq 0$. Now expand this out using the Cartan formula. If $0 \leq i < m/2p$ then $2(n-1) > 2pn - m - 2(p-1)(n-i)$ and so by the unstable axiom for the action on homology, $P^{n-i}(z) = 0$. If $m/2p \leq i \leq m/2(p-1)$ then similarly $P^i(x) = 0$. So the only possibility is that $P^i(x) \neq 0$ for some $i > m/2(p-1)$, so that $\deg(P^i(x)) > 0$. This completes the proof of the converse. We leave to the reader the easy task of rewriting this part of the proof for $p = 2$. □

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